Contraction semigroups on $L_\infty(\mathbf{R})$

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Dedicated to the memory of Günter Lumer 1929–2005

Abstract

If X is a non-degenerate vector field on $\mathbf{R}$ and $H = -X^2$ we examine conditions for the closure of H to generate a continuous semigroup on L_∞ which extends to the L_p-spaces. We give an example which cannot be extended and an example which extends but for which the real part of the generator on L_2 is not lower semibounded.

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1 Introduction

The Lumer–Phillips theorem [LuP] is a cornerstone of the theory of continuous semigroups. The theorem characterizes the generator of a contraction semigroup with the aid of a dissipativity condition. The latter is based on the elementary properties of the operator $-d^2/dx^2$ of double differentiation acting on $C_0(\mathbf{R})$. In this note we analyze contraction semigroups S generated by squares $-X^2$ of vector fields $X = a\,d/dx$ acting on $C_0(\mathbf{R})$, or $L_\infty(\mathbf{R})$. An integral part of the analysis consists of examining the one-parameter groups T generated by X. Throughout we assume $a > 0$. If a is smooth this is the one-dimensional analogue of Hörmander's condition [Hör].

First, we identify the kernel of S acting on $L_\infty(\mathbf{R})$. Secondly, T is defined as a weak* continuous group of contractions on L_∞ and we derive necessary and sufficient conditions for it to extend to a continuous group on the $L_p(\mathbf{R}\,;\rho\,dx)$-spaces with $p \in [1,\infty\rangle$, where $\rho:\mathbf{R} \to \langle 0,\infty\rangle$ is a C^∞-function. These conditions also ensure that S extends to a continuous semigroup. Thirdly, we characterize those S, or T, which extend to a contraction semigroup, or group, on $L_p(\mathbf{R}\,;\rho\,dx)$ for some $p \in [1,\infty\rangle$. Fourthly, we give an example of a smooth vector field with a uniformly bounded coefficient for which neither T nor S can be extended to any of the L_p-spaces with $p < \infty$. Fifthly, we give an example of a smooth vector field with a uniformly bounded coefficient which is uniformly bounded away from zero for which T and S extend to all the L_p-spaces but the real part of the generator of S on $L_2(\mathbf{R}\,;\rho\,dx)$ is not lower semibounded. In particular the L_2-generator cannot satisfy a Gårding inequality. Since the Gårding inequality is the usual starting point for the analysis of elliptic divergence form operators on $L_2(\mathbf{R}\,;\rho\,dx)$, e.g., operators of the form X^*X, this example clearly demonstrates that the theory of 'non-divergent' form operators such as $-X^2$ on $L_\infty(\mathbf{R})$ is very different. Finally we discuss the volume doubling property for balls (intervals) whose radius (length) is measured by the distance associated with X.

2 Preliminaries

Let $a:\mathbf{R} \to \langle 0,\infty\rangle$ be a locally bounded differentiable function and assume the derivative a' is locally bounded. Further assume

$$\int_0^\infty dx\, a(x)^{-1} = \infty = \int_{-\infty}^0 dx\, a(x)^{-1} \quad . \tag{1}$$

Equip $\mathbf{R}$ with the measure $\rho\,dx$ where $\rho:\mathbf{R} \to \langle 0,\infty\rangle$ is a C^∞-function. Consider the vector field $X = a\,d/dx$ and the corresponding operators X_{min} and X_{max} on $L_\infty(\mathbf{R}\,;\rho\,dx)$ with domains $D(X_{\mathrm{min}}) = C_c^\infty(\mathbf{R})$ and $D(X_{\mathrm{max}}) = C_c^1(\mathbf{R})$. Set $H_{\mathrm{min}} = -X_{\mathrm{min}}^2$ and $H_{\mathrm{max}} = -X_{\mathrm{max}}^2$. Since we are dealing with operators on L_∞ it is appropriate to deal with the weak* topology.

Proposition 2.1

I. *The operators X_{min} and X_{max} are weak* closable and $\overline{X}_{\mathrm{min}} = \overline{X}_{\mathrm{max}}$, where the bar denotes the weak* closure.*

II. *The operator H_{max} is weak* closable and its weak* closure $\overline{H}_{\mathrm{max}}$ generates a semigroup S which is weak* continuous, positive, contractive and holomorphic in the open right half-plane.*

III. $\overline{H}_{\max} = -\overline{X}_{\max}^{\,2}$ *and in particular* $\overline{X}_{\max}^{\,2}$ *is weak* closed.*

IV. *If* $a \in C^\infty(\mathbf{R})$ *then* $\overline{H}_{\min} = \overline{H}_{\max}$*, where* $\overline{H}_{\min}$ *is the weak* closure of* $H_{\min}$.

Proof For all $x_0 \in \mathbf{R}$ the ordinary differential equation $\dot{x} = a(x)$, with initial data $x(0) = x_0$, has a unique maximal solution which we denote by $t \mapsto e^{tX}x_0$. Since a satisfies (1) this maximal solution is defined for all $t \in \mathbf{R}$. Moreover, $e^{sX}e^{tX}x_0 = e^{(s+t)X}x_0$ and

$$\int_{x_0}^{e^{tX}x_0} dx\, a(x)^{-1} = t \tag{2}$$

for all $s, t \in \mathbf{R}$ and $x_0 \in \mathbf{R}$. In addition both the maps $t \mapsto e^{tX}x_0$ and $x \mapsto e^{sX}x$ are continuous. In particular for all $t \in \mathbf{R}$ the map $T_t \colon L_\infty \to L_\infty$ defined by $(T_t\varphi)(y) = \varphi(e^{-tX}y)$ is an isometry and T is a weak* continuous group on L_∞. This group is automatically positive and we next show that its generator is the weak* closure of the operator $X_{\min}$ on L_∞.

Clearly $X_{\min} \subseteq X_{\max}$ and by a standard regularization argument it follows that $\overline{X}_{\min} = \overline{X}_{\max}$. Hence to simplify notation we now set $X_0 = \overline{X}_{\min} = \overline{X}_{\max}$.

One computes from (2) that

$$\frac{d}{dy}\, e^{tX}y = \frac{a(e^{tX}y)}{a(y)}$$

for all $t \in \mathbf{R}$ and $y \in \mathbf{R}$. Therefore

$$\frac{d}{dy}(T_t\varphi)(y) = \varphi'(e^{-tX}y) \cdot \frac{a(e^{tX}y)}{a(y)}$$

for all $\varphi \in D(X_{\max})$, $y \in \mathbf{R}$ and $t > 0$. So $T_t(D(X_{\max})) \subseteq D(X_{\max})$ for all $t > 0$. Moreover,

$$\begin{aligned} t^{-1}(\varphi - T_t\varphi)(y) &= -t^{-1}\int_0^t ds\, \frac{d}{ds}\varphi(e^{-sX}y) \\ &= t^{-1}\int_0^t ds\, \varphi'(e^{-sX}y)\, a(e^{-sX}y) = t^{-1}\int_0^t ds\, (T_sX_{\max}\varphi)(y) \end{aligned}$$

for all $\varphi \in D(X_{\max})$, $t > 0$ and $y \in \mathbf{R}$, since φ' is continuous. So $\lim_{t\to 0} t^{-1}(I - T_t)\varphi = X_{\max}\varphi$ strongly in L_∞ and $X_{\max}$ is the restriction of the generator of T. Since $D(X_{\max})$ is invariant under T and weak* dense it follows from Corollary 3.1.7 of [BrR] that $X_0 = \overline{X}_{\max}$ is the generator of T.

Next define the semigroup S by the integral algorithm

$$S_t = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2(4t)^{-1}}T_s \quad . \tag{3}$$

Obviously S is weak* continuous, positive, contractive and holomorphic in the open right half-plane. Let H_0 denote the weak* closed generator of S. If $\varphi \in D(X_0^{\,2})$ then

$$\begin{aligned} t^{-1}\,(I - S_t)\varphi &= t^{-1}\,(4\pi t)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2(4t)^{-1}}(I - T_s)\varphi \\ &= t^{-1}\,(4\pi t)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2(4t)^{-1}}\int_0^s du\,(s-u)\,T_u\,X_0^{\,2}\varphi \\ &= (4\pi)^{-1/2}\int_{-\infty}^{\infty} ds\, e^{-s^2/4}\int_0^s du\,(s-u)\,T_{t^{1/2}u}\,X_0^{\,2}\varphi \end{aligned}$$

and it follows in the weak* limit $t \to 0$ that $\varphi \in D(H_0)$. Hence $H_0 \supseteq -X_0^{\,2}$. To prove $H_0 = -X_0^{\,2}$ it suffices to establish that the range $R(I - X_0^{\,2})$ of $I - X_0^{\,2}$ is equal to L_∞. But X_0 generates the continuous group T. Therefore $R(I \pm X_0) = L_\infty$. Moreover, $I - X_0^{\,2} = (I - X_0)(I + X_0)$. Hence $R(I - X_0^{\,2}) = L_\infty$ and $H_0 = -X_0^{\,2}$.

Clearly $H_{\max} \subseteq -X_0^{\,2} = H_0$ so $H_{\max}$ is weak* closable. It remains to prove that the weak* closure $\overline{H}_{\max}$ of $H_{\max}$ is equal to H_0.

Since $T_tD(X_{\max}) \subseteq D(X_{\max})$ and $X_{\max}T_t\varphi = T_tX_{\max}\varphi$ for all $\varphi \in D(X_{\max})$ one deduces by iteration that $T_tD(X_{\max}^2) \subseteq D(X_{\max}^2)$ and $X_{\max}^2T_t\varphi = T_tX_{\max}^2\varphi$ for all $\varphi \in D(X_{\max}^2)$. Next it follows from (3), by a Riemann approximation argument, that $S_tD(X_{\max}^2) \subseteq D(\overline{X_{\max}^2})$ and $\overline{X_{\max}^2}S_t\varphi = S_tX_{\max}^2\varphi$ for all $\varphi \in D(X_{\max}^2)$ and all $t > 0$. Since S_t is continuous it further follows that $S_tD(\overline{X_{\max}^2}) \subseteq D(\overline{X_{\max}^2})$ for all $t > 0$. But $C_c^1(\mathbf{R}) \subseteq D(X_{\max}^2) \subseteq D(\overline{H}_{\max})$ is weak* dense in L_∞ by the assumed differentiability of a. Hence by Corollary 3.1.7 of [BrR] it follows that $D(\overline{H}_{\max})$ is a core of H_0. Therefore $\overline{H}_{\max} = H_0$.

Finally, if $a \in C^\infty(\mathbf{R})$ then $C_c^\infty(\mathbf{R})$ is a core for $X_{\max}^2$. Therefore $\overline{H}_{\min} \supseteq H_{\max}$. Since $H_{\min} \subseteq H_{\max}$ this completes the proof of the proposition. □

Remark 2.2 It follows by definition that $T_tC_0(\mathbf{R}) \subseteq C_0(\mathbf{R})$ for all $t \in \mathbf{R}$ and a simple estimate shows that the restriction of T to $C_0(\mathbf{R})$ is strongly continuous. Therefore $S_tC_0(\mathbf{R}) \subseteq C_0(\mathbf{R})$ for all $t > 0$ and the restriction of S to $C_0(\mathbf{R})$ is also strongly continuous. This is a direct consequence of the algorithm (3). Thus T is a Feller group and S is a Feller semigroup. Now let X_{00} and H_{00} denote the generators of the restricted group and the restricted semigroup, respectively. Then a slight modification of the foregoing argument allows one to obtain similar characterizations of the generators but in terms of norm closures. For example, X_{00} is the norm closure of $X_{\min}$ which is equal to the norm closure of $X_{\max}$. The discussion of H_{00} can in fact be simplified. Since X_{00} generates a strongly continuous group of isometries the operator $-X_{00}^2$ is dissipative in the sense of Lumer and Phillips [LuP] and it is norm closed by standard estimates (see, for example, [Rob] Lemma III.3.3). But one again has $R(I \pm X_{00}) = L_\infty$. Therefore $R(I - X_{00}^2) = L_\infty$. Then $-X_{00}^2$ generates a strongly continuous contraction semigroup by the Lumer–Phillips theorem and it follows by uniqueness that $H_{00} = -X_{00}^2$.

One can associate a distance with the vector field X by the definition

$$d(x\,;y) = \sup\{|\psi(x) - \psi(y)|\,;\, \psi \in C_c^\infty(\mathbf{R})\,,\, \|X\psi\|_\infty \leq 1\,\} \quad . \tag{4}$$

Clearly one has

$$|\psi(x) - \psi(y)| = \Big|\int_x^y dz\,\psi'(z)\Big| \leq \Big|\int_x^y dz\,a(z)^{-1}\Big|$$

for all $\psi \in C_c^\infty(\mathbf{R})$ with $\|X_{\min}\psi\|_\infty \leq 1$. So

$$d(x\,;y) \leq \Big|\int_x^y dz\,a(z)^{-1}\Big| \quad .$$

But by regularizing a^{-1} on a compact interval one deduces that the inequality is in fact an equality, i.e.,

$$d(x\,;y) = \Big|\int_x^y dz\,a(z)^{-1}\Big|$$

for all $x, y \in \mathbf{R}$. Note that by setting $x = e^{-sX}y$ and using (2) one finds

$$d(e^{-sX}y\,;y) = \Big| \int_y^{e^{-sX}y} dz\, a(z)^{-1} \Big| = |s| \quad . \tag{5}$$

Therefore the distance is invariant under the flow in the sense that

$$d(e^{-tX}x\,;e^{-tX}y) = d(x\,;y)$$

for all $x, y \in \mathbf{R}$ and all $t \geq 0$. This follows by setting $x = e^{-sX}y$ and

$$d(e^{-tX}x\,;e^{-tX}y) = d(e^{-sX}e^{-tX}y\,;e^{-tX}y) = |s| = d(e^{-sX}y\,;y) = d(x\,;y) \quad ,$$

where we have used (5).

Now one can calculate the kernel of the semigroup S.

Proposition 2.3 *The kernel K of the semigroup S on $L_\infty(\mathbf{R})$ is given by*

$$K_t(x\,;y) = (4\pi t)^{-1/2}\,(a(y)\rho(y))^{-1} e^{-d(x;y)^2(4t)^{-1}} \tag{6}$$

for all $x, y \in \mathbf{R}$ and $t > 0$. Moreover, K_t is continuous and $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $x \in \mathbf{R}$.

Proof First by (3) one has

$$(S_t\varphi)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds\, e^{-s^2(4t)^{-1}} \varphi(e^{-sX}x)$$

for all $\varphi \in C_c^\infty(\mathbf{R})$, $t > 0$ and $x \in \mathbf{R}$. Therefore by a change of variables $y = e^{-sX}x$ one deduces that

$$(S_t\varphi)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} dy\, a(y)^{-1} e^{-d(x;y)^2(4t)^{-1}} \varphi(y)$$

since $|s| = d(x\,;y)$ by (5). The representation (6) follows immediately.

Clearly K_t is continuous and $H_{\max}\mathbb{1} = 0$. So $S_t\mathbb{1} = \mathbb{1}$ in L_∞-sense. Therefore $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $t > 0$ and almost every $x \in \mathbf{R}$. Moreover, the map $x \mapsto \int dy\,\rho(y)\,K_t(x\,;y)$ is continuous. Hence $\int dy\,\rho(y)\,K_t(x\,;y) = 1$ for all $t > 0$ and $x \in \mathbf{R}$. □

3 Extension properties

Although T is defined as a group of isometries and S as a contraction semigroup on L_∞ they do not automatically extend to the L_p-spaces. This requires extra boundedness conditions on the coefficient function a and the density function ρ. The following proposition gives necessary and sufficient conditions for T to extend to a continuous group and sufficient conditions for S to extend to a continuous semigroup.

Proposition 3.1 *Let T be the group of isometries of $L_\infty(\mathbf{R}\,;\rho\,dx)$ defined by $(T_t\varphi)(y) = \varphi(e^{-tX}y)$. The following conditions are equivalent for all $C \geq 1$ and $\omega \geq 0$.*

I. *There is a* $p \in [1,\infty\rangle$ *such that* T *extends to a (strongly) continuous group on* $L_p(\mathbf{R}\,;\rho\,dx)$ *satisfying the bounds* $\|T_t\|_{p\to p} \leq C^{1/p}\,e^{\omega|t|/p}$ *for all* $t \in \mathbf{R}$*.*

II. *For all* $p \in [1,\infty\rangle$ *the group* T *extends to a (strongly) continuous group on* $L_p(\mathbf{R}\,;\rho\,dx)$ *satisfying the bounds* $\|T_t\|_{p\to p} \leq C^{1/p}\,e^{\omega|t|/p}$ *for all* $t \in \mathbf{R}$*.*

III. $$a(y)\rho(y) \leq C\,e^{\omega d(x;y)}\,a(x)\rho(x) \qquad \text{for all } x, y \in \mathbf{R}.$$

Moreover, if these conditions are satisfied then the semigroup S *extends to a (strongly) continuous semigroup on all the* L_p*-spaces,* $p \in [1,\infty\rangle$*, satisfying the bounds*

$$\|S_t\|_{p\to p} \leq \Big((2\,C)^{1/p}\,e^{\omega^2 t/p}\Big) \wedge \Big(2\,C^{1/p}\,e^{\omega^2 t/p^2}\Big)$$

if $\omega > 0$ *and* $\|S_t\|_{p\to p} \leq C^{1/p}$ *if* $\omega = 0$*, for all* $t > 0$*.*

Proof First assume Condition I is satisfied. Then for all $\varphi \in L_p$ one has

$$\|T_t\varphi\|_p^p = \int_{\mathbf{R}} dy\,\rho(y)\,|\varphi(e^{-tX}y)|^p \quad .$$

Secondly, by a change of variables $x = e^{-tX}y$ one finds

$$\|T_t\varphi\|_p^p = \int_{\mathbf{R}} dx\,\frac{a(e^{tX}x)}{a(x)}\,\rho(e^{tX}x)\,|\varphi(x)|^p = \int_{\mathbf{R}} dx\,\rho(x)\,\Big(\frac{a(e^{tX}x)\rho(e^{tX}x)}{a(x)\rho(x)}\Big)\,|\varphi(x)|^p \quad .$$

Therefore

$$\sup_{x\in\mathbf{R}} \Big(\frac{a(e^{tX}x)\rho(e^{tX}x)}{a(x)\rho(x)}\Big)^{1/p} = \|T_t\|_{p\to p} \leq C^{1/p}e^{\omega|t|/p}$$

for all $t \in \mathbf{R}$ and $x \in \mathbf{R}$. Hence

$$a(e^{tX}x)\rho(e^{tX}x) \leq C\,e^{\omega|t|}a(x)\rho(x)$$

for all $t \in \mathbf{R}$ and $x \in \mathbf{R}$. Setting $y = e^{tX}x$ and noting that $d(x\,;y) = |t|$ one deduces that Condition III is satisfied. Conversely, the same calculation shows that if Condition III is satisfied then

$$\|T_t\varphi\|_p \leq C^{1/p}e^{\omega|t|/p}\|\varphi\|_p \tag{7}$$

for all $p \in [1,\infty\rangle$, $\varphi \in L_p$ and $t \in \mathbf{R}$. In addition if $\varphi \in C_c^\infty$ then one calculates that

$$\varphi - T_t\varphi = \int_0^t ds\,T_s X_{\min}\varphi \quad .$$

Hence using (7) and the density of C_c^∞ in L_p one concludes that T_t extends to a continuous semigroup on L_p satisfying the bounds (7), i.e., Condition II is valid. The implication II⇒III is trivial.

If the conditions are satisfied then S extends to the L_p-spaces by (3). The estimates on the norms of S_t are established in two steps. First, if $\omega > 0$ then it follows from (3) and the estimates on $\|T_s\|_{1\to 1}$ that

$$\|S_t\|_{1\to 1} \leq 2\,C\,e^{\omega^2 t}$$

for all $t > 0$. Since S is contractive on L_∞ one deduces from interpolation that

$$\|S_t\|_{p\to p} \leq (2\,C)^{1/p}\,e^{\omega^2 t/p}$$

for all $p \in \langle 1, \infty\rangle$ and $t > 0$. Alternatively, one can reverse the reasoning and use the interpolated bounds $\|T_s\|_{p\to p} \leq C^{1/p}\, e^{\omega|s|/p}$ together with (3) to calculate that

$$\|S_t\|_{p\to p} \leq 2\, C^{1/p}\, e^{\omega^2 t/p^2}$$

for all $p \in [1, \infty]$ and $t > 0$.

If $\omega = 0$ similar arguments apply and both lead to the bounds $\|S_t\|_{p\to p} \leq C^{1/p}$. □

The situation described by the proposition simplifies if $C = 1$. Then Condition III together with (5) implies that

$$\begin{aligned}\pm(a\rho)'(y)\, a(y) &= \lim_{t\downarrow 0} t^{-1}\Big((a\rho)(e^{\pm tX}y) - (a\rho)(y)\Big)\\ &\leq \limsup_{t\downarrow 0} t^{-1}(e^{\omega t} - 1)(a\rho)(y) = \omega\,(a\rho)(y)\end{aligned}$$

for all $y \in \mathbf{R}$. Thus $\|\rho^{-1}(a\rho)'\|_\infty \leq \omega$. Conversely, if $\|\rho^{-1}(a\rho)'\|_\infty \leq \omega$ then

$$\rho(e^{tX}y)^{-1}\frac{d}{dt}\Big(e^{-\omega t}\,(a\rho)(e^{\pm tX}y)\Big) \leq 0$$

for all $t \geq 0$. Hence Condition III is satisfied with $C = 1$. But the condition $\|\rho^{-1}(a\rho)'\|_\infty \leq \omega$ can be expressed in terms of the vector field. Therefore one has the following corollary.

Corollary 3.2 *The following conditions are equivalent for all $\omega \geq 0$.*

I. *There is a $p \in [1, \infty\rangle$ such that T extends to a continuous group on $L_p(\mathbf{R}\,; \rho\, dx)$ satisfying the bounds $\|T_t\|_{p\to p} \leq e^{\omega|t|/p}$ for all $t \in \mathbf{R}$.*

II. *For all $p \in [1, \infty\rangle$ the group T extends to a continuous group on $L_p(\mathbf{R}\,; \rho\, dx)$ satisfying the bounds $\|T_t\|_{p\to p} \leq e^{\omega|t|/p}$ for all $t \in \mathbf{R}$.*

III. $$\|\rho^{-1}(a\rho)'\|_\infty \leq \omega\quad.$$

IV. $$|(\psi, (X + X^*)\varphi)| \leq \omega\, \|\psi\|_q\, \|\varphi\|_p \qquad \textit{for all } \varphi, \psi \in C_c^\infty(\mathbf{R})$$
and for one pair (for all pairs) of dual exponents $p, q \in [1, \infty]$.

Moreover, if these conditions are satisfied then the semigroup S extends to a continuous semigroup on all the L_p-spaces, $p \in [1, \infty\rangle$, satisfying the bounds

$$\|S_t\|_{p\to p} \leq e^{\omega^2 t/p^2}$$

for all $t > 0$. In addition $H_{\max}$ satisfies a Gårding inequality. Precisely,

$$\mathrm{Re}(\varphi, H_{\max}\varphi) \geq (1 - \varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1})\|X + X^*\|_{2\to 2}^2\|\varphi\|_2^2$$

for all $\varphi \in C_c^\infty(\mathbf{R})$ and $\varepsilon > 0$.

Proof The equivalence of the first three conditions and the existence of the extension of the semigroup S follow from Proposition 2.1 and the above discussion. Conditions III and IV are equivalent because

$$\begin{aligned}(\psi, X\varphi) + (X\psi, \varphi) &= \int_{\mathbf{R}} dx\, (a\rho)(x)\Big(\psi(x)\, \varphi'(x) + \psi'(x)\, \varphi(x)\Big)\\ &= \int_{\mathbf{R}} dx\, \rho(x)\Big(\rho(x)^{-1}(a\rho)'(x)\Big)\psi(x)\, \varphi(x)\end{aligned}$$

for all $\varphi, \psi \in C_c^\infty(\mathbf{R})$. It remains to prove the Gårding inequality.

If $\varepsilon > 0$ then

$$\begin{aligned}
\mathrm{Re}(\varphi, H_{\max}\varphi) &= -\,\mathrm{Re}(X^*\varphi, X\varphi) \\
&= \|X\varphi\|_2^2 - \mathrm{Re}((X^* + X)\varphi, X\varphi) \\
&\geq \|X\varphi\|_2^2 - \|(X^* + X)\varphi\|_2 \|X\varphi\|_2 \\
&\geq (1-\varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1}\|X + X^*\|_{2\to 2}^2 \|\varphi\|_2^2
\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbf{R})$. $\square$

The corollary, applied with $\omega = 0$, gives the following criteria for T or S to extend to a contraction group or semigroup on the L_p-spaces.

Proposition 3.3 *The following are equivalent.*

- **I.** *There is a $p \in [1, \infty\rangle$ such that T extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*
- **II.** *For all $p \in [1, \infty\rangle$ the group T extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*
- **III.** *There is a $p \in [1, \infty\rangle$ such that S extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*
- **IV.** *For all $p \in [1, \infty\rangle$ the semigroup S extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$.*
- **V.** *The function $a\rho$ is constant.*

Proof The implications V$\Leftrightarrow$I$\Leftrightarrow$II$\Rightarrow$IV follow from Corollary 3.2 and the implication IV$\Rightarrow$III is trivial.

The proof of the implication III$\Rightarrow$V relies on the reasoning of Lumer and Phillips.

If Condition III is valid for some $p \in [1, 2]$ then it follows by interpolation with the contraction semigroup on L_∞ that Condition III is valid for all $p > 2$. Hence it suffices to show that if $p \in \langle 2, \infty\rangle$ and S extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$ then the function $a\rho$ is constant, i.e., Condition V is valid. Fix $p \in \langle 2, \infty\rangle$ and assume S extends to a continuous contraction group on $L_p(\mathbf{R}\,; \rho\, dx)$. Then it follows from the Lumer–Phillips theorem, [LuP] Theorem 3.1, that the generator H of the semigroup S on $L_p(\mathbf{R}\,; \rho\, dx)$ is dissipative. So if $[\,\cdot\,,\,\cdot\,]$ is a semi-inner product on $L_p(\mathbf{R}\,; \rho\, dx)$ then $\mathrm{Re}[H\varphi, \varphi] \geq 0$ for all $\varphi \in D(H)$. If $\varphi \in C_c^2(\mathbf{R})$ is real valued then $\varphi \in D(H_{\max})$ and $H_{\max}\varphi \in L_p(\mathbf{R}\,; \rho\, dx)$. So $\varphi \in D(H)$ and $H_{\max}\varphi = H\varphi$. Moreover,

$$\int d(a\,\rho\,\varphi^{p-1})\, a\,(d\,\varphi) = \int \rho\,\varphi^{p-1}\, H_{\max}\varphi = \int \rho\,\varphi^{p-1}\, H\varphi = \|\varphi\|_p^{p-2}[H\varphi, \varphi] \geq 0$$

where $d = d/dx$. Hence

$$\int d(a\,\rho\,\varphi^{p-1})\, a\,(d\,\varphi) \geq 0 \tag{8}$$

for all real valued $\varphi \in W_c^{1,\infty}(\mathbf{R})$ by approximation.

Next fix $\tau \in C_c^\infty(\mathbf{R})$ such that $0 \leq \tau \leq 1$, $\tau(0) = 1$ and τ is decreasing on $[0, \infty\rangle$. For all $n \in \mathbf{N}$ define $\varphi_n \in W_c^{1,\infty}(\mathbf{R})$ by

$$\varphi_n = (a\rho)^{-1/p}\,(\tau \circ \Phi_n)$$

where

$$\Phi_n(x) = n^{-1}\,d(0\,;x)^2 = n^{-1}\Big(\int_0^x a^{-1}\Big)^2 \quad .$$

Then

$$\varphi_n'(x) = -p^{-1}(a\rho)(x)^{-1-p^{-1}}\,(a\rho)'(x)\,\tau(\Phi_n(x)) \\ + 2n^{-1}(a\rho)(x)^{-1/p}\,\tau'(\Phi_n(x))\Big(\int_0^x a^{-1}\Big)a(x)^{-1}$$

and

$$(a\rho\,\varphi_n')(x) = -p^{-1}(a\rho)(x)^{-1/p}\,(a\rho)'(x)\,\tau(\Phi_n(x)) \\ + 2n^{-1}\rho(x)\,(a\rho)(x)^{-1/p}\,\tau'(\Phi_n(x))\Big(\int_0^x a^{-1}\Big) \quad .$$

Similarly, $(a\rho\,\varphi_n^{p-1})(x) = (a\rho)(x)^{1/p}\,\tau(\Phi_n(x))^{p-1}$ and

$$(a\rho\,\varphi_n)'(x) = p^{-1}(a\rho)(x)^{-1+p^{-1}}\,(a\rho)'(x)\,\tau(\Phi_n(x))^{p-1} \\ + 2n^{-1}(p-1)\rho(x)\,(a\rho)(x)^{-1+p^{-1}}\,\tau(\Phi_n(x))^{p-2}\,\tau'(\Phi_n(x))\Big(\int_0^x a^{-1}\Big) \quad .$$

Then by (8) it follows that

$$0 \le \int \rho^{-1} d(a\rho\,\varphi_n^{p-1})\,a\rho\,(d\,\varphi_n) \\ = \int dx\Big(- p^{-2}\rho(x)^{-1}\,(a\rho)(x)^{-1}\,(a\rho)'(x)^2\Big(\tau(\Phi_n(x))\Big)^2 \\ - 2n^{-1}(1-2p^{-1})\,(a\rho)(x)^{-1}\,(a\rho)'(x)\,\tau(\Phi_n(x))^{p-1}\,\tau'(\Phi_n(x))\Big(\int_0^x a^{-1}\Big) \\ + 4n^{-2}(p-1)\rho(x)\,(a\rho)(x)^{-1}\tau(\Phi_n(x))^{p-1}\,\Big(\tau'(\Phi_n(x))\Big)^2 d(0\,;x)^2\Big) \quad .$$

Using the estimate $a\,b \le \varepsilon a^2 + (4\varepsilon)^{-1}b^2$ for the second term, setting $\varepsilon = (2p(p-2))^{-1}$ and rearranging one finds

$$(2p^2)^{-1}\int \rho^{-1}\,(a\rho)^{-1}\,((a\rho)')^2(\tau\circ\Phi_n)^2 \\ \le n^{-1}\int \rho\,(a\rho)^{-1}\Big(4(p-1)(\tau\circ\Phi_n)^{p-2} + 2(p-2)^2(\tau\circ\Phi_n)^{2p-2}\Big)(\tau'\circ\Phi_n))^2\,\Phi_n \tag{9}$$

for all $n \in \mathbf{N}$. There are $b, c > 0$ such that

$$y\,\Big(4(p-1)\tau(y)^{p-2} + 2(p-2)^2\tau(y)^{2p-2}\Big)(\tau'(y))^2 \le c\,e^{-(4b)^{-1}y}$$

for all $y \in [0,\infty\rangle$. Then

$$\Big((a\rho)^{-1}\Big(4(p-1)(\tau\circ\Phi_n)^{p-2} + 2(p-2)^2(\tau\circ\Phi_n)^{2p-2}\Big)(\tau'\circ\Phi_n)^2\,\Phi_n\Big)(x) \\ \le c\,(a\rho)(x)^{-1}\,e^{-d(0;x)^2(4bn)^{-1}} \\ = c\,(4\pi\,b\,n)^{1/2}\,K_{bn}(0\,;x)$$

uniformly for all $x \in \mathbf{R}$ and $n \in \mathbf{N}$. Using Proposition 2.3 one deduces that

$$\int \rho\,(a\rho)^{-1}\Big(4(p-1)(\tau\circ\Phi_n)^{p-2} + 2(p-2)^2(\tau\circ\Phi_n)^{2p-2}\Big)(\tau'\circ\Phi_n)^2\,\Phi_n \le c\,(4\pi\,b\,n)^{1/2}$$

for all $n \in \mathbf{N}$. Finally (9) and the monotone convergence theorem establishes that

$$\begin{aligned}(2p^2)^{-1}\int \rho^{-1}\,(a\rho)^{-1}\Big((a\rho)'\Big)^2 &= \lim_{n\to\infty}(2p^2)^{-1}\int \rho^{-1}\,(a\rho)^{-1}\Big((a\rho)'\Big)^2(\tau\circ\Phi_n)^2\\ &\le \lim_{n\to\infty} n^{-1}\,c\,(4\pi\,b\,n)^{1/2} = 0 \quad .\end{aligned}$$

Therefore $(a\rho)' = 0$ as required. □

In the unweighted case, i.e., $\rho = 1$, the proposition establishes that S extends to a contraction semigroups on one of the L_p-spaces with $p < \infty$ only in the case that X is proportional to d/dx.

4 Examples

Next we give two examples of rather unexpected properties although there is nothing inherently pathological about the weight ρ or the coefficient a. In fact in both examples $\rho = 1$ and the coefficient a of the vector field is strictly positive, smooth and uniformly bounded. The first example gives a continuous group T and semigroup S which do not extend from L_∞ to the other L_p spaces. The principal reason for this singular behaviour is the fact that $\inf a = 0$, i.e., there is a mild degeneracy at infinity.

Example 4.1 Let $\rho = 1$. For all $n \in \mathbf{N}_0$ define $h_n = n!^{-1}$. Define $y_n \in \mathbf{R}$ for all $n \in \mathbf{N}_0$ by $y_0 = 0$ and inductively

$$y_{n+1} = y_n + 4^{-1}(h_n + h_{n+1}) + 2^{-1}$$

for all $n \in \mathbf{N}$. Define $\tilde{a}\colon \mathbf{R} \to \langle 0,\infty\rangle$ by

$$\tilde{a}(x) = \begin{cases} h_n & \text{if } x \in [y_n - 4^{-1}h_n, y_n + 4^{-1}h_n\rangle \quad (n \in \mathbf{N}_0)\quad, \\ 1 & \text{if } x \in [y_n + 4^{-1}h_n, y_n + 4^{-1}h_n + 2^{-1}\rangle \quad (n \in \mathbf{N}_0)\quad, \\ 1 & \text{if } x \in \langle -\infty, 0] \quad . \end{cases}$$

Then $\tilde{a}(y_n) = h_n$ and $\int_{y_n}^{y_{n+1}} dx\,\tilde{a}(x)^{-1} = 1$ for all $n \in \mathbf{N}$. Next we regularize $\tilde{a}^{-1}$. For all $n \in \mathbf{N}_0$ let $\chi_n \in C_c^\infty(\mathbf{R})$ be such that $\chi_n \ge 0$, $\int \chi_n = 1$, $\operatorname{supp}\chi_n \subseteq [-8^{-1}h_n, 8^{-1}h_n]$ and $\chi_n(-x) = \chi_n(x)$ for all $x \in \mathbf{R}$. Define $a \in C^\infty(\mathbf{R})$ by

$$a(x)^{-1} = \begin{cases} (\chi_0 * \tilde{a}^{-1})(x) & \text{if } x \le 0 \quad, \\ (\chi_n * \tilde{a}^{-1})(x) & \text{if } n \in \mathbf{N}_0 \text{ and } x \in [y_n - 4^{-1}h_n - 4^{-1}, y_n + 4^{-1}h_n + 4^{-1}\rangle \quad . \end{cases}$$

Then $a(y) = h_n$ for all $y \in [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n]$ and $\int_{y_n}^{y_{n+1}} dx\,a(x)^{-1} = 1$ for all $n \in \mathbf{N}$. Hence $d(y_n\,;y_{n+1}) = 1$ for all $n \in \mathbf{N}$. But $a(y_n) = (n+1)\,a(y_{n+1})$ for all $n \in \mathbf{N}$. Therefore Condition III of Proposition 3.1 is not valid. In particular the group T does not extend to

any of the other L_p spaces. Next we show that the semigroup S also does not extend to another L_p space.

Let $p \in [1, \infty\rangle$, $t > 0$ and let q be the dual exponent of p. For all $n \in \mathbf{N}$ set $I_n = [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n]$. Let $n \in \mathbf{N}$. Set $\varphi = \mathbb{1}_{I_{n+1}}$ and $\psi = \mathbb{1}_{I_n}$. Then $\|\varphi\|_p = |I_{n+1}|^{1/p}$ and $\|\psi\|_q = |I_n|^{1/q}$. Moreover,

$$
\begin{aligned}
(\psi, S_t\varphi) &= (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy\, a(y)^{-1}\, e^{-d(x;y)^2(4t)^{-1}} \\
&\geq (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy\, a(y)^{-1}\, e^{-3d(x;y)^2 t^{-1}} \\
&= (4\pi t)^{-1/2} |I_n|\, |I_{n+1}|\, h_{n+1}^{-1}\, e^{-3d(x;y)^2 t^{-1}} \quad .
\end{aligned}
$$

So

$$
\|S_t\|_{p\to p} \geq (4\pi t)^{-1/2} |I_n|^{1/p}\, |I_{n+1}|^{1/q}\, h_{n+1}^{-1}\, e^{-3d(x;y)^2 t^{-1}} = (64\pi t)^{-1/2}(n+1)^{1/p} \quad .
$$

Hence the operator S_t on L_∞ does not extend to a continuous operator on L_p for any $p \in [1, \infty\rangle$ or $t > 0$. □

In the next example the coefficient a of X is uniformly bounded above and below by a positive constant but $\sup a' = \infty$ The semigroup S extends to a continuous semigroup on all the L_p-spaces but the real part of the generator of S on L_2 is not lower semibounded. This contrasts with the case of continuous self-adjoint semigroups where boundedness of the semigroup immediately implies lower semiboundedness of the generator.

Example 4.2 First, let $\rho = 1$ and let $\chi \in C_c^\infty(\mathbf{R})$ be such that $0 \leq \chi \leq 3$, $\chi' \geq 0$, $\chi(x) = 0$ if $x \leq 0$, $\chi(x) = 3$ if $x \geq 3$ and $\chi(x) = x$ if $1 \leq x \leq 2$. Define $a \colon \mathbf{R} \to [1, 4]$ by

$$
a(x) = 1 + \sum_{n=1}^{\infty} \Big(\chi(n(x - 16n)) - \chi(n(x - (16n + 8)) \Big) \quad .
$$

Thus $a = 1$ on an infinite sequence of intervals of length almost equal to 8 spaced at distance 8 one from the other. On the intermediate intervals a increases smoothly to the value 4 and then decreases in a similar fashion to the value 1. The rate of increase and decrease, however, becomes larger with the distance of the interval from the origin. Nevertheless $a \in C^\infty(\mathbf{R})$ and the bounds of Proposition 3.1.III are valid with $C = 4$ and $\omega = 0$. In particular S_t extends to the L_p-spaces and $\|S_t\|_{p\to p} \leq 4^{1/p}$.

Secondly, let $n \in \mathbf{N}$ with $n \geq 4$. Let $\psi \in C^\infty(\mathbf{R})$ be such that $\psi(x) = 3$ for all $x \leq 16n + 8$, $0 \leq \psi' \leq n^{1/2}$, $\psi'(x) = 0$ for all $x \geq 16n + 8 + 4n^{-1}$ and $\psi'(x) = n^{1/2}$ for all $x \in [16n + 8 + n^{-1}, 16n + 8 + 2n^{-1}]$. Then $3 \leq \psi(16n + 8 + 4n^{-1}) \leq 5$. Now define $\varphi \in C_c^\infty(\mathbf{R})$ by

$$
\varphi(x) = \begin{cases} \chi(x - (16n + 4)) & \text{if } x \leq 16n + 8 \\ \psi(x) & \text{if } x \in [16n + 8, 16n + 8 + 4n^{-1}] \\ 3^{-1}\psi(16n + 8 + 4n^{-1})\Big(3 - \chi(x - (16n + 8 + 4n^{-1})\Big) & \text{if } x \geq 16n + 8 + 4n^{-1} \end{cases}
$$

Then $\|\varphi\|_2 \le 5 \cdot (12)^{1/2} = (300)^{1/2}$ and

$$\|\varphi'\|_2 \le 2\|\chi'\|_\infty + n^{1/2}(4n^{-1})^{1/2} + 3^{-1}\psi(16n+8+4n^{-1})\|\chi'\|_\infty \le 2 + 4\|\chi'\|_\infty \quad .$$

But $a'\,a\,\varphi\,\varphi' \le 0$ and

$$-(a'\varphi, X\varphi) \ge \int_{16n+8+n^{-1}}^{16n+8+2n^{-1}} (-a'\,a\,\varphi\,\varphi') \ge \int_{16n+8+n^{-1}}^{16n+8+2n^{-1}} n \cdot 2 \cdot 3 \cdot n^{1/2} = 6n^{1/2}$$

by the previous estimates. Therefore

$$\begin{aligned}\mathrm{Re}(\varphi, H_{\min}\varphi) &= \|X\varphi\|_2^2 + \mathrm{Re}(a'\varphi, X\varphi)\\ &\le \|a\|_\infty^2(2+4\|\chi'\|_\infty)^2 - 8n^{1/2} \le -300^{-1}\Big(6n^{1/2} - 16(2+4\|\chi'\|_\infty)^2\Big)\|\varphi\|_2^2 \quad .\end{aligned}$$

Consequently, $\mathrm{Re}\, H_{\min}$ is not lower semibounded. This is despite the uniform boundedness of S on L_2.

Next, since S is uniformly bounded on each of the L_p-spaces, the spectrum $\sigma(H)$ of the generator H of the semigroup on L_p is contained in the right half-plane. But $a(x) \in [1,4]$ for all $x \in \mathbf{R}$. Therefore $4^{-1}|x-y| \le d(x\,;y) \le |x-y|$ and Proposition 2.3 implies that

$$K_t(x\,;y) \le (4\pi t)^{-1/2}\, e^{-|x-y|^2(64t)^{-1}}$$

for all $x, y \in \mathbf{R}$ and $t > 0$. Hence it follows from [Kun] or [LiV] that $\sigma(H)$ is independent of $p \in [1,\infty]$. On the other hand $\mathrm{Re}\, H_{\min}$ is not lower semibounded on L_2 and the above estimates establish that $\langle -\infty, 0] \subset \Theta(H)$, the L_2-numerical range of H. Therefore $\Theta(H) \ne \sigma(H)$ on L_2.

In fact this example illustrates the extreme situation that the spectrum of H is contained in the right half plane but the numerical range is the whole complex plane. This follows since one can establish that the numerical range $\Theta(H) = \mathbf{C}$ by a small modification of the foregoing estimates applied to the function $\tilde\varphi \in C_c^\infty(\mathbf{R})$ defined by

$$\tilde\varphi(x) = e^{i\lambda x}\,\tau(x) + \varphi(x) \quad ,$$

where $\lambda \in \mathbf{R}$ and $\tau \in C_c^\infty(\langle -1, 4\rangle)$ is fixed such that $0 \le \tau \le 1$ and $\tau|_{[0,3]} = 1$. One also uses the observation that the numerical range is convex.

Finally note that the semigroup S has a bounded holomorphic extension to the open right half-plane on each of the L_p-spaces, $p \in [1,\infty\rangle$. This follows from the explicit form of the kernel given in Propositions 2.3. Therefore the operator H is of type S_{0+}. Nevertheless, since $\Theta(H) = \mathbf{C}$ the operator H is not sectorial. $\Box$

5 Volume doubling

Let $V(x\,;r)$ denote the measure of the ball of radius r centred at x, i.e., the set $\{y : d(x\,;y) < r\} = \langle e^{-rX}x, e^{rX}x\rangle$. Then V is defined, as usual, to have the volume doubling property if there is a $c > 0$ such that

$$V(x\,;2r) \le c\, V(x\,;r)$$

for all $r > 0$. This property can be immediately related to the conditions of Proposition 3.1 which are necessary and sufficient for the continuous extension of T to the L_p-spaces.

Proposition 5.1

I. *If the equivalent conditions of Proposition 3.1 are satisfied then*

$$V(x\,;2r) \le 2\,C^2\,e^{3\omega}\,V(x\,;r) \tag{10}$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ where C and ω are the parameters of Proposition 3.1. Moreover if $\omega = 0$ then (10) is valid for all $x \in \mathbf{R}$ and $r > 0$.

II. *If there exist $c > 0$ and a function $v\colon \langle 0,\infty\rangle \to \mathbf{R}$ such that*

$$c^{-1}\,v(r) \le V(x\,;r) \le c\,v(r)$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ then Condition III *of Proposition 3.1 is satisfied with $\omega = 0$.*

Proof It follows by definition that

$$V(x\,;r) = \int_{e^{-rX}x}^{e^{rX}x} dy\,\rho(y) \quad .$$

But

$$\frac{d}{dr}V(x\,;r) = (a\rho)(e^{rX}x) + (a\rho)(e^{-rX}x) \quad .$$

Hence

$$V(x\,;r) = \int_0^r ds\,\Big((a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x)\Big) = \int_{-r}^r ds\,(a\rho)(e^{sX}x) \quad .$$

Therefore if Condition III of Proposition 3.1 is satisfied one estimates that

$$2\,C^{-1}r\,e^{-\omega r}(a\rho)(x) \le V(x\,;r) \le 2\,C\,r\,e^{\omega r}(a\rho)(x)$$

for all $x \in \mathbf{R}$ and $r > 0$. These bounds imply (10) for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$ or, if $\omega = 0$, for all $r > 0$.

If, however, the assumptions of the second statement are valid then

$$c^{-1}\,v(r) \le V(x\,;r) = \int_0^r ds\,(a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x) \le r \max_{y\in[e^{-X}x,e^{X}x]}(a\rho)(y)$$

for all $x \in \mathbf{R}$ and $r \in \langle 0,1]$. Similarly

$$c\,v(r) \ge r \min_{y\in[e^{-X}x,e^{X}x]}(a\rho)(y) \quad .$$

Hence there exists a $c_1 > 0$ such that $c_1^{-1}\,r \le v(r) \le c_1\,r$ for all $r \in \langle 0,1]$. But then

$$\begin{aligned} 2(a\rho)(x) &= \lim_{r\downarrow 0} r^{-1}\int_0^r ds\,(a\rho)(e^{sX}x) + (a\rho)(e^{-sX}x) \\ &= \lim_{r\downarrow 0} r^{-1}\,V(x\,;r) \le \limsup_{r\downarrow 0} r^{-1}\,c\,v(r) \le c\,c_1 \end{aligned}$$

for all $x \in \mathbf{R}$. Similarly $2(a\rho)(x) \ge (c\,c_1)^{-1}$. Hence $(2c\,c_1)^{-1} \le a\rho \le 2^{-1}c\,c_1$ and Condition III of Proposition 3.1 is satisfied with $\omega = 0$. $\Box$

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